A Hamiltonian action of the Schrödinger-Virasoro algebra on a space of periodic time-dependent Schrödinger operators in $(1+1)$-dimensions

Claude Roger, Jeremie Unterberger

February 3, 2009

A Hamiltonian action of the Schrödinger-Virasoro algebra on a space of periodic time-dependent Schrödinger operators in $(1+1)$-dimensions

Claude Roger[a] and **Jérémie Unterberger**[b]

[a]Institut Camille Jordan ,[1] Université Claude Bernard Lyon I,
21 avenue Claude Bernard, 69622 Villeurbanne Cedex, France

[b]Institut Elie Cartan,[2] Université Henri Poincaré Nancy I,
B.P. 239, F – 54506 Vandœuvre lès Nancy Cedex, France

Abstract

Let $\mathcal{S}^{aff} := \{-2i\partial_t - \partial_r 2 + V(t,r) \mid V \in C^\infty(\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R})\}$ be the space of Schrödinger operators in $(1+1)$-dimensions with periodic time-dependent potential. The action on $\mathcal{S}^{aff}$ of a large infinite-dimensional reparametrization group SV [5, 7], called the Schrödinger-Virasoro group and containing the Virasoro group, is proved to be Hamiltonian for a certain symplectic structure on $\mathcal{S}^{aff}$. More precisely, the action of SV appears to be a projected coadjoint action of a group of pseudo-differential symbols, G, of which SV is a quotient, while the symplectic structure is inherited from the corresponding Kirillov-Kostant-Souriau form.

[1]Laboratoire associé au CNRS UMR 5208
[2]Laboratoire associé au CNRS UMR 7502

0 Introduction

The Schrödinger-Virasoro Lie algebra $\mathfrak{sv}$ was originally introduced in Henkel[2] as a natural infinite-dimensional extension of the Schrödinger algebra. Recall the latter is defined as the algebra of projective Lie symmetries of the free Schrödinger equation in (1+1)-dimensions

$$(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r 2)\psi(t,r) = 0. \tag{0.1}$$

These act on equation (0.1) as the following first-order operators

$$\begin{aligned} L_n &= -t^{n+1}\partial_t - \frac{1}{2}(n+1)t^n r\partial_r + \frac{\mathrm{i}}{4}\mathcal{M}(n+1)nt^{n-1}r2 - (n+1)\lambda t^n \\ Y_m &= -t^{m+\frac{1}{2}}\partial_r + \mathrm{i}\mathcal{M}(m+\frac{1}{2})t^{m-\frac{1}{2}}r \\ M_p &= \mathrm{i}\mathcal{M}t^p \end{aligned} \tag{0.2}$$

with $\lambda = 1/4$ and $n = 0, \pm 1$, $m = \pm\frac{1}{2}$, $p = 0$. The 0th-order terms in (0.2) correspond on the group level to the multiplication of the wave function by a phase. To be explicit, the 6-dimensional *Schrödinger group* $\mathcal{S}$ acts on ψ by the following transformations

$$(L_{-1}, L_0, L_1) \ : \quad \psi(t,r) \to \psi'(t',r') = (ct+d)^{-1/2} e^{-\frac{1}{2}\mathrm{i}\mathcal{M}cr2/(ct+d)}\psi(t,r) \tag{0.3}$$

where $t' = \frac{at+b}{ct+d}$, $r' = \frac{r}{ct+d}$ with $ad - bc = 1$;

$$(Y_{\pm\frac{1}{2}}) \ : \quad \psi(t,r) \to \psi(t,r') = e^{-\mathrm{i}\mathcal{M}((vt+r_0)(r-v/2)}\psi(t,r) \tag{0.4}$$

where $r' = r - vt - r_0$;

$$(M_0) \ : \quad \psi(t,r) \to e^{\mathrm{i}\mathcal{M}\gamma}\psi(t,r). \tag{0.5}$$

The Schrödinger group is isomorphic to a semi-direct product of $SL(2,\mathbb{R})$ (corresponding to time-reparametrizations (0.3)) by the Heisenberg group $\mathcal{H}_1$ (corresponding to the Galilei transformations (0.4), (0.5)). Note that the last transformation (0.5) (multiplication by a constant phase) is generated by the commutators of the Galilei transformations (0.4) - these do not commute because of the added phase terms, which produce a central extension.

The free Schrödinger equation comes out naturally when considering many kinds of problems in out-of-equilibrium statistical physics. Its analogue in equilibrium statistical physics is the Laplace equation $\Delta\psi = 0$. In two-dimensional space, the latter equation is invariant by local conformal transformations which generate (up to a change of variables) the well-known (centerless) Virasoro algebra $\mathrm{Vect}(S1)$, otherwise known as the Lie algebra of C^∞-vector fields on the torus $S1 := \{e^{\mathrm{i}\theta}, \theta \in [0, 2\pi]\}$. There is no substitute for $\mathrm{Vect}(S1)$ when time-dependence is included, but the *Schrödinger-Virasoro Lie algebra*

$$\mathfrak{sv} \simeq \langle L_n, Y_m, M_p \mid n, p \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z}\rangle \tag{0.6}$$

shares some properties with it. First, the Lie subalgebra $\mathrm{span}(L_n,\ n \in \mathbb{Z})$ is isomorphic to $\mathrm{Vect}(S1)$. Actually, $\mathfrak{sv}$ is isomorphic to a semi-direct product $\mathrm{Vect}(S1)$ by a nilpotent Lie algebra.

Second, there exists a natural action of the Schrödinger-Virasoro group SV integrating $\mathfrak{sv}$ (see [5]) on an affine space $\mathcal{S}^{aff} := \{-2i\partial_t - \partial_r 2 + V(t,r)\}$ of Schrödinger operators with time-periodic potential, which generalizes the well-known action $\phi_* : \partial_t 2 + u(t) \to \partial_t 2 + (\dot{\phi}(t))2(u \circ \phi)(t) + \frac{1}{2}\Theta(\phi)(t)$ (Θ=Schwarzian derivative, see below) of the Virasoro group on Hill operators. The infinitesimal action of $\mathfrak{sv}$ on $\mathcal{S}^{aff}$, denoted by $d\sigma_{1/4}$, is introduced in section 1. It is essentially obtained by conjugating Schrödinger operators with the above functional transformations (0.2).

The main result of this paper is the following.

Theorem.

There exists a Poisson structure on $\mathcal{S}^{aff} = \{-2i\partial_t - \partial_r 2 + V(t,r)\}$ for which the infinitesimal action $d\sigma_{1/4}$ of $\mathfrak{sv}$ is Hamiltonian.

The analogue in the case of Hill operators is well-known (see for instance [1]). Namely, the action of the Virasoro group on the space $\mathcal{H}$ of Hill operators is equivalent to its affine coadjoint action with central charge $c = \frac{1}{2}$, with the identification $\partial_t 2 + u(t) \to u(t)dt2 \in \mathfrak{vir}^*_{\frac{1}{2}}$, where $\mathfrak{vir}^*_c$ is the affine hyperplane $\{(X,c) \mid X \in (\mathrm{Vect}(S1))^*\}$. Hence this action preserves the canonical KKS (Kirillov-Kostant-Souriau) structure on $\mathfrak{vir}^*_{\frac{1}{2}} \simeq \mathcal{H}$. As well-known, one may exhibit a bi-Hamiltonian structure on $\mathfrak{vir}^*$ which provides an integrable system on $\mathcal{H}$ associated to the Korteweg-De Vries equation.

The above identification does not hold true any more in the case of the Schrödinger action of SV on the space of Schrödinger operators, which is *not* equivalent to its coadjoint action (see [5], section 3.2). Hence the existence of a Poisson structure for which the action on Schrödinger operators is Hamiltonian has to be proved in the first place. It turns out that the action on Schrödinger operators is more or less the restriction of the coadjoint action of a much larger group G on its dual. The `Lie algebra of` G is introduced in Definition 5.2. The way we went until we came across this Lie algebra $\mathfrak{g}$ is a bit tortuous. The first idea (as explained in [3]) was to see $\mathfrak{sv}$ as a *subquotient* of an algebra $D\Psi D$ of *extended* pseudodifferential symbols on the line: one checks easily that the assignment $\mathcal{L}_f \to f(\xi)\partial_\xi$, $\mathcal{Y}_g \to g(\xi)\partial_\xi^{\frac{1}{2}}$, $\mathcal{M}_h \to h(\xi)$ gives a linear application $\mathfrak{sv} \to D\Psi D := \mathbb{R}[\xi,\xi^{-1}]]\ [\partial_\xi^{\frac{1}{2}}, \partial_\xi^{-\frac{1}{2}}]]$ which respects the Lie brackets of both Lie algebras, up to unpleasant terms which are pseudodifferential symbols with negative order. Define $D\Psi D_{\leq\kappa}$ as the subspace of pseudodifferential symbols with order $\leq \kappa$. Then $D\Psi D_{\leq 1}$ is a Lie subalgebra of $D\Psi D$, $D\Psi D_{\leq -\frac{1}{2}}$ is an ideal, and the above assignment defines an isomorphism $\mathfrak{sv} \simeq D\Psi D_{\leq 1}/D\Psi D_{\leq -\frac{1}{2}}$.

The second idea (sketched in [6]) was to use a non-local transformation $\Theta : D\Psi D \to \Psi D$ (ΨD being the usual algebra of pseudo-differential symbols) which maps $\partial_\xi^{\frac{1}{2}}$ to ∂_r and ξ to $\frac{1}{2}r\partial_r^{-1}$ (see Definition 2.4). The transformation Θ is formally an integral operator, simply associated to the heat kernel, which maps the first-order differential operator $-2i\mathcal{M}\partial_t - \partial_\xi$ into $-2i\mathcal{M}\partial_t - \partial_r 2$. The operator $-2i\mathcal{M}\partial_t - \partial_\xi$ (which is simply the $\partial_{\bar{z}}$-operator in light-cone coordinates) is now easily seen to be invariant under an infinite-dimensional Lie algebra which generates (as an associative algebra) an algebra isomorphic to $D\Psi D$. One has thus defined a natural action of $D\Psi D$ on the space of solutions of the free Schrödinger equation $(-2i\mathcal{M}\partial_t - \partial_r 2)\psi = 0$.

The crucial point now is that (after conjugation with Θ, i.e. coming back to the usual (t,r)-coordinates) the action of $D\Psi D_{\leq 1}$ coincides *up to pseudodifferential symbols of negative order* with the above realization (0.2) of the generators L_n, Y_m, M_p $(n,p \in \mathbb{Z}, m \in \frac{1}{2}+\mathbb{Z})$. In other words, loosely speaking, the abstract isomorphism $\mathfrak{sv} \simeq D\Psi D_{\leq 1}/D\Psi D_{\leq -\frac{1}{2}}$ has received a concrete interpretation, and one has somehow reduced a problem concerning *differential operators in two variables* t,r into a problem concerning *time-dependent pseudodifferential operators in one variable*, which is a priori much simpler.

Integrable systems associated to Poisson structures on the loop algebra $\mathfrak{L}_t(\Psi D)$ over ΨD (with the usual Kac-Moody cocycle $(X,Y) \to \oint K(\dot{X}(t), Y(t)\ dt$, where K is Adler's trace on ΨD) have been studied by A. Reiman ???. In our case computations show that the $\mathfrak{sv}$-action on Schrödinger operators is related to the coadjoint action of $\mathfrak{L}_t((\Psi D_r)_{\leq 1})$, where $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})$ is a central extension of $\mathfrak{L}_t((\Psi D_r)_{\leq 1})$ which is unrelated to the Kac-Moody cocycle.

Actually, the above scheme works out perfectly fine only for the restriction of the $\mathfrak{sv}$-action to the nilpotent part of $\mathfrak{sv}$. For reasons explained in sections 3 and 4, the generators of $\mathrm{Vect}(S1) \subset \mathfrak{sv}$ play a particular rôle. So the action $d\sigma_{1/4}$ of $\mathfrak{sv}$ is really obtained through the *projection* on the second component of the coadjoint action of an extended Lie algebra $\mathfrak{g} := \mathrm{Vect}(S1) \ltimes \mathfrak{L}_t(\widetilde{(\Psi D_r)}_{\leq 1})$. The definition of $\mathfrak{g}$ requires in itself some work and is given only at the end of section 5.

It is natural to expect that there should exist some bi-Hamiltonian structure on $\mathcal{S}^{aff}$ allowing to define some unknown integrable system. We hope to answer this question in the future.

Note that the action $d\sigma_{1/4}$ restricted to the (stable) affine subspace $\mathcal{S}^{aff}_{\leq 2} := \{-2\mathrm{i}\mathcal{M}\partial_t - \partial_r 2 + V_2(t)r2 + V_1(t)r + V_0(t)\}$ has been shown in [7] to be Hamiltonian for a totally different Poisson structure. The two constructions are unrelated.

Here is the outline of the article. The definitions and results from [5] needed on the Schrödinger-Virasoro algebra and its action on Schrödinger operators are briefly recalled in section 1. Section 2 on pseudo-differential operators is mainly introductive, except for the definition of the non-local transformation Θ. The realization of $D\Psi D_{\leq 1}$ as symmetries of the free Schrödinger equation is explained in section 3. Sections 4 and 5 are devoted to the construction of the extended Lie algebra $LPsiDr$ and its extension $\mathfrak{g}$. Finally, the projected coadjoint action of $\mathfrak{g}$ is defined in section 6, and the action $d\sigma_{1/4}$ of $\mathfrak{sv}$ on Schrödinger operators is obtained as a restriction of this action to a stable submanifold in section 7.

Notation: In the sequel, the derivative with respect to r, resp. t will always be denoted by a prime ($'$), resp. by a dot, namely, $V'(t,r) := \partial_r V(t,r)$ and $\dot{V}(t,r) := \partial_t V(t,r)$ (except the third-order time derivative $\frac{d3V}{dt3}$, for typographical reasons).

1 Definition of the action of $\mathfrak{sv}$ on Schrödinger operators

We recall in this preliminary section the properties of the Schrödinger-Virasoro algebra $\mathfrak{sv}$ proved in [5] that will be needed throughout the article.

We shall denote by $\mathrm{Vect}(S1)$ the Lie algebra of 2π-periodic C^∞-vector fields. It is generated by $(\ell_n;\ n \in \mathbb{Z})$, $\ell_n := \mathrm{i}e^{\mathrm{i}n\theta}\partial_\theta$, with the following Lie brackets: $[\ell_n, \ell_p] = (n-p)\ell_{n+p}$. Setting $t = e^{\mathrm{i}\theta} \in S1$, one has $\ell_n = -t^{n+1}\partial_t$. It may be seen as the Lie algebra of $\mathrm{Diff}(S1)$, which is the group of orientation-preserving smooth diffeomorphisms of the torus.

For any $\lambda \in \mathbb{R}$, $\mathrm{Diff}(S1)$ admits a representation on the space of $(-\lambda)$-*densities*

$$\mathcal{F}_\lambda := \{f(\theta)(d\theta)^{-\lambda}, \quad f \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})\}$$

defined by the natural action by change of variables,

$$\pi_\lambda(\phi^{-1})f = (\dot{\phi})^{-\lambda} f \circ \phi.$$

Definition 1.1 (Schrödinger-Virasoro algebra) *(see [5], Definition 1.2)*

We denote by $\mathfrak{sv}$ *the Lie algebra with generators* $L_n, Y_m, M_n (n \in \mathbb{Z}, m \in \frac{1}{2}+\mathbb{Z})$ *and following relations (where* $n, p \in \mathbb{Z}, m, m' \in \frac{1}{2}+\mathbb{Z}$*) :*

$$[L_n, L_p] = (n-p)L_{n+p}$$
$$[L_n, Y_m] = (\frac{n}{2} - m)Y_{n+m}, \quad [L_n, M_p] = -pM_{n+p};$$
$$[Y_m, Y_{m'}] = (m-m')M_{m+m'},$$
$$[Y_m, M_p] = 0, \quad [M_n, M_p] = 0.$$

If f *(resp.* g*,* h*) is a Laurent series,* $f = \sum_{n\in\mathbb{Z}} f_n t^{n+1}$*, resp.* $g = \sum_{n\in\kappa+\mathbb{Z}} g_n t^{n+\frac{1}{2}}$*,* $h = \sum_{n\in\mathbb{Z}} h_n t^n$*, then we shall write*

$$\mathcal{L}_f = \sum f_n L_n, \quad \mathcal{Y}_g = \sum g_n Y_n, \quad \mathcal{M}_h = \sum h_n M_n. \tag{1.1}$$

Let $\mathfrak{g}_0$ =span$(L_n,\ n \in \mathbb{Z})$ and $\mathfrak{h}$ =span$(Y_m, M_p,\ m \in \frac{1}{2}+\mathbb{Z}, p \in \mathbb{Z})$. Then $\mathfrak{g}_0 \simeq \mathrm{Vect}(S1)$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{sv}$, and $\mathfrak{sv} \simeq \mathfrak{g}_0 \ltimes \mathfrak{h}$ enjoys a semi-direct structure. Note also that $\mathfrak{h}$ is pronilpotent.

The Schrödinger-Virasoro algebra may be exponentiated into a group $SV = G_0 \ltimes H$, where $G_0 \simeq \mathrm{Diff}(S1)$ and H is a nilpotent Lie group (see [5], Theorem 1.4).

Definition 1.2 *(see [5], Definition 1.3)*

Denote by $d\pi_\lambda$ *the representation of* $\mathfrak{sv}$ *as differential operators of order one on* $\mathbb{R}2$ *with coordinates* t, r *defined by*

$$d\pi_\lambda(\mathcal{L}_f) = -f(t)\partial_t - \frac{1}{2}\dot{f}(t)r\partial_r + \frac{1}{4}\mathrm{i}\mathcal{M}\ddot{f}(t)r2 - \lambda\dot{f}(t)$$
$$d\pi_\lambda(\mathcal{Y}_g) = -g(t)\partial_r + \mathrm{i}\mathcal{M}\dot{g}(t)r$$
$$d\pi_\lambda(\mathcal{M}_h) = \mathrm{i}\mathcal{M}h(t) \tag{1.2}$$

Note that $d\pi_\lambda(L_n)$, $d\pi_\lambda(Y_m)$, $d\pi_\lambda(M_p)$ coincide with the formulas (0.2) given in the Introduction.

The infinitesimal representation $d\pi_\lambda$ of $\mathfrak{sv}$ may be exponentiated into a representation π_λ of the group SV (see [5], Proposition 1.6). We shall not need explicit formulas in this article. Let us simply write the action of time-diffeomorphisms:

$$(\pi_\lambda(\phi;0)f)(t',r') = (\dot{\phi}(t))^{-\lambda} e^{\frac{1}{4}\mathrm{i}\frac{\ddot{\phi}(t)}{\dot{\phi}(t)}r2} f(t,r) \tag{1.3}$$

if $\phi \in G_0 \simeq \mathrm{Diff}(S1)$ induces the coordinate change $(t,r) \to (t',r') = (\phi(t), r\sqrt{\dot{\phi}(t)})$. It appears clearly in this formula that the parameter λ is a 'scaling dimension' or the weight of a density.

Let us now introduce the manifold $\mathcal{S}^{aff}$ of Schrödinger operators we want to consider.

Definition 1.3 (Schrödinger operators) *(see [5], Definition 2.1)*

Let $\mathcal{S}^{lin}$ be the vector space of second order operators on $\mathbb{R}2$ defined by

$$D \in \mathcal{S}^{lin} \Leftrightarrow D = h(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r 2) + V(t,r), \quad h, V \in C^\infty(\mathbb{R}2)$$

and $\mathcal{S}^{aff} \subset \mathcal{S}^{lin}$ the affine subspace of 'Schrödinger operators' given by the hyperplane $h = 1$.

In other words, an element of $\mathcal{S}^{aff}$ is the sum of the free Schrödinger operator $-2\mathrm{i}\mathcal{M}\partial_t - \partial_r 2$ and of a potential V.

The action of SV on Schrödinger operators is essentially the conjugate action of $\pi_{1/4}$:

Proposition 1.4 *(see [5], Proposition 2.5, Proposition 2.6)*

1. *Let $\sigma_{1/4} : SV \to Hom(\mathcal{S}^{lin}, \mathcal{S}^{lin})$ the representation of the group of SV on the space of Schrödinger operators defined by the left-and-right action*

$$\sigma_{1/4}(g) : D \to \pi_{5/4}(g) D \pi_{1/4}(g)^{-1}, \quad g \in SV, D \in \mathcal{S}^{lin}.$$

Then $\sigma_{1/4}$ restricts to an affine action on the affine subspace $\mathcal{S}^{aff}$ which is given by the following formulas:

$$\begin{aligned} \sigma_{1/4}(\phi;0).(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r 2 + V(t,r)) = \\ -2\mathrm{i}\partial_t - \partial_r 2 + \phi'(t)V(\phi(t), r\sqrt{\phi'(t)}) + \frac{1}{2} r2\Theta(\phi)(t) \end{aligned} \tag{1.4}$$

$$\begin{aligned} \sigma_{1/4}(1;(a,b)).(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r 2 + V(t,r)) = \\ -2\mathrm{i}\mathcal{M}\partial_t - \partial_r 2 + V(t, r - a(t)) - 2ra''(t) - (2b'(t) - a(t)a''(t)). \end{aligned} \tag{1.5}$$

where $\Theta : \phi \to \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right) 2$ is the Schwarzian derivative.

2. The infinitesimal action $d\sigma_{1/4} : X \to \frac{d}{dt}|_{t=0} \left(\sigma_{1/4}(\exp tX)\right)$ of $\mathfrak{sv}$ writes (see notations in the Introduction) :

$$d\sigma_{1/4}(\mathcal{L}_f)(V) = -f\dot{V} - \frac{1}{2}\dot{f}(rV' + 2V) + \frac{1}{8}r2\frac{d3V}{dt3}$$
$$d\sigma_{1/4}(\mathcal{Y}_g)(V) = -g\dot{V} - \frac{1}{2}\ddot{g}r$$
$$d\sigma_{1/4}(\mathcal{M}_h)(V) = -\dot{h} \tag{1.6}$$

This action by reparametrization has been studied elsewhere [7]. Once restricted to the stable submanifold $\mathcal{S}^{aff}_{\leq 2} := \{-2\mathrm{i}\partial_t - \partial_r 2 + V_0(t) + V_1(t)r + V_2(t)r2\}$ of Schrödinger operators with time-dependent *quadratic* potential, it exhibits a rich variety of finite-codimensional orbits, whose classification is obtained by generalizing classical results due to A. A. Kirillov on orbits of Hill operators under the Virasoro group. Also, a parametrization of operators by their stabilizers yields a natural symplectic structure for which the $\sigma_{1/4}$-action is Hamiltonian. These ideas do not carry over to the whole space $\mathcal{S}^{aff}$ whose symplectic structure will be obtained below by a totally different method.

2 Algebras of pseudodifferential symbols

Definition 2.1 (algebra of formal pseudodifferential symbols) *Let $\Psi D := \mathbb{R}[z, z^{-1}]]\ [\partial_z, \partial_z^{-1}]]$ be the associative algebra of Laurent series in z, ∂_z with defining relation $[\partial_z, z] = 1$.*

Using the coordinate $z = e^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, one may see elements of ΨD as formal pseudodifferential operators with periodic coefficients.

The algebra ΨD comes with a trace, called *Adler's trace*, defined in the Fourier coordinate θ by

$$\mathrm{Tr}\left(\sum_{q=-\infty}^{N} f_q(\theta)\partial_\theta^q\right) = \int_0^{2\pi} f_{-1}(\theta)\ d\theta. \tag{2.1}$$

Coming back to the coordinate z, this is equivalent to setting

$$\mathrm{Tr}(a(z)\partial_z^q) = \delta_{q,-1} \cdot \frac{1}{2\mathrm{i}\pi}\oint a(z)dz \tag{2.2}$$

where $\frac{1}{2\mathrm{i}\pi}\oint$ is the Cauchy integral giving the residue a_{-1} of the Laurent series $\sum_{p=-\infty}^{N} a_p z^p$.

For any $n \leq 1$, the vector subspace generated by the pseudo-differential operators $D = f_n(z)\partial_z^n + f_{n-1}(z)\partial_z^{n-1} + \ldots$ of degree $\leq n$ is a Lie subalgebra of ΨD that we shall denote by $\Psi D_{\leq n}$. We shall sometimes write $D = O(\partial_z^n)$ for a pseudodifferential operator of degree $\leq n$. Also, letting $OD = \Psi D_{\geq 0} = \{\sum_{k=0}^{n} f_k(z)\partial_z^k, \quad n \geq 0\}$ (differential operators) and $\mathfrak{volt} = \Psi D_{\leq -1}$

(called: *Volterra algebra*), we shall denote by (D_+, D_-) the decomposition of $D \in \Psi D$ along the direct sum $OD \oplus \mathfrak{volt}$.

We shall also need the following 'extended' algebra of formal pseudodifferential symbols.

Definition 2.2 (algebra of extended pseudodifferential symbols) *Let $D\Psi D$ be the extended pseudo-differential algebra generated as an associative algebra by ξ, ξ^{-1} and $\partial_\xi^{\frac{1}{2}}, \partial_\xi^{-\frac{1}{2}}$.*

Let $D \in D\Psi D$. As in the case of the usual algebra of pseudodifferential symbols, we shall write $D = O(\partial_z^\kappa)$ ($\kappa \in \frac{1}{2}\mathbb{Z}$) for an extended pseudodifferential symbol with degree $\leq \kappa$, and denote by $D\Psi D_{\leq\kappa}$ the Lie subalgebra $\mathrm{span}(f_j(\xi)\partial_\xi^j \ ; \ j = \kappa, \kappa - \frac{1}{2}, \kappa - 1, \ldots)$ if $\kappa \leq 1$.

The Lie algebra $D\Psi D$ contains two interesting subalgebras for our purposes:

(i) $\mathrm{span}(f_1(\xi)\partial_\xi, f_0(\xi); \ f_1, f_0 \in C^\infty(S1))$ which is isomorphic to $\mathrm{Vect}(S1) \ltimes \mathcal{F}_0$;

(ii) $D\Psi D_{\leq 1}$:=$\mathrm{span}(f_\kappa(\xi)\partial_\xi^\kappa; \ \kappa = 1, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, f_\kappa \in C^\infty(S1))$ which is the Lie algebra generated by $\mathrm{span}(f_1(\xi)\partial_\xi, f_{\frac{1}{2}}(\xi)\partial_\xi^{1/2}, f_0(\xi); \ f_1, f_{\frac{1}{2}}, f_0 \in C^\infty(S1))$.

The Lie algebra $\Psi D_{\leq 1}$ may be integrated to a group in the following way. Consider first the pronilpotent Lie group Volt := $\exp \mathfrak{volt} = \{1 + f_{-1}(\xi)\partial_\xi^{-1} + \ldots\}$ obtained by the formal exponentiation of pseudo-differential symbols, $\exp V = \sum_{k\geq 0} \frac{V^k}{k!}$, $V \in \mathfrak{volt}$. It is easily extended to the semi-direct product group $\overline{\mathrm{Volt}} = \exp \mathcal{F}_0 \ltimes \mathrm{Volt}$ (where $\exp \mathcal{F}_0 = \exp C^\infty(S1) \simeq \{f \in C^\infty(S1) \mid \forall\theta \in [0, 2\pi], f(e^{i\theta}) \neq 0\}$) which integrates $\underline{\Psi D_{\leq 0}} \simeq \mathcal{F}_0 \ltimes \mathfrak{volt}$. Finally, $\mathrm{Diff}(S1)$ acts naturally on $\overline{\mathrm{Volt}}$, which yields a Lie group $\mathrm{Diff}(S1) \ltimes \overline{\mathrm{Volt}}$ integrating $\Psi D_{\leq 1} \simeq \mathrm{Vect}(S1) \ltimes \Psi D_{\leq 0}$.

This explicit construction does not work for $D\Psi D_{\leq 1}$ because the formal series $\sum_{k\geq 0} \frac{V^k}{k!}$ is not in $C^\infty[\xi, \xi^{-1}]] \ [\partial_\xi, \partial_\xi^{-1}]]$ if $V = f_{1/2}(\xi)\partial_\xi^{\frac{1}{2}} + O(\partial_\xi 0)$, $f_{1/2} \not\equiv 0$. Yet the Campbell-Hausdorff formula makes it possible to integrate $D\Psi D_{\leq 1}$ by a similar procedure into an abstract group $DG_{\leq 1}$:

Lemma 2.3 *The Lie algebra $D\Psi D_{\leq 1}$ may be exponentiated into a group $DG_{\leq 1}$.*

Proof.

First exponentiate $D\Psi D_{\leq 1/2}$ =$\mathrm{span}(f_\kappa(\xi)\partial_\xi^\kappa \ ; \ \kappa = \frac{1}{2}, 0, -\frac{1}{2}, \ldots)$ by defining $DG_{\leq\frac{1}{2}}$:= $\exp D\Psi D_{\leq\frac{1}{2}}$ with multiplication given by the Campbell-Hausdorff formula

$$\exp\left(f(\xi)\partial_\xi^{\frac{1}{2}} + D_1\right)\exp\left(g(\xi)\partial_\xi^{\frac{1}{2}} + D_2\right)$$
$$= \exp\left\{\left((f(\xi) + g(\xi))\partial_\xi^{\frac{1}{2}} + D_1 + D_2 + \ldots\right) + \frac{1}{2}\left[f(\xi)\partial_\xi^{\frac{1}{2}} + D_1, g(\xi)\partial_\xi^{\frac{1}{2}} + D_2\right] + \ldots\right\} \tag{2.3}$$

$(D_1, D_2 \in D\Psi D_{\leq 0})$; the first Lie bracket is $D\Psi D_{\leq 0}$-valued, and the successive iterated brackets belong to $D\Psi D_{\leq \kappa_1}$, $D\Psi D_{\leq \kappa_2}$, ... where $(\kappa_n)_{n\in\mathbb{N}^*}$ is a strictly decreasing sequence (with $\kappa_1 = -\frac{1}{2}$), hence the series converges.

Then define the semi-direct product $DG_{\leq 1} := \mathrm{Diff}(S1) \ltimes DG_{\leq \frac{1}{2}}$ by the following natural action ρ of $\mathrm{Diff}(S1)$ on $DG_{\leq \frac{1}{2}}$:

– let $\rho' : \mathrm{Diff}(S1) \to Lin(D\Psi D_{\leq \frac{1}{2}})$ be the linear action defined by

$$\rho'(\phi)(f\partial_\xi^\kappa) = (f \circ \phi^{-1}) \,.\, (\phi' \circ \phi^{-1} \,.\, \partial_\xi)^\kappa, \quad \kappa \leq \frac{1}{2}$$

where $(\phi' \circ \phi^{-1} \,.\, \partial_\xi)^{\frac{1}{2}} = \sqrt{\phi' \circ \phi^{-1}}\partial_\xi^{\frac{1}{2}} + \ldots$ is the usual square root of operators (recall $\phi' > 0$ by definition), and

$$\begin{aligned}
(\phi' \circ \phi^{-1} \,.\, \partial_\xi)^\kappa &= \left[(\phi' \circ \phi^{-1} \,.\, \partial_\xi)^{\frac{1}{2}}\right]^{2\kappa} \\
&= \left(\partial_\xi^{-\frac{1}{2}}(\phi' \circ \phi^{-1})^{-\frac{1}{2}} \,.\, (1 + \partial_\xi^{-\frac{1}{2}}(\phi' \circ \phi^{-1})^{-\frac{1}{2}} O(\partial_\xi^{-\frac{1}{2}}))^{-1}\right)^{-2\kappa} \\
&\in O(\partial_\xi^\kappa) \qquad (2.4)
\end{aligned}$$

if $\kappa \leq 0$;

– if $\phi \in \mathrm{Diff}(S1)$, one lets $\rho(\phi)\exp D := \exp(\rho'(\phi)D) \in DG_{\leq \frac{1}{2}}$.

□

It turns out that a certain non-local transformation gives an isomorphism between $D\Psi D$ and ΨD. For the sake of the reader, we shall in the sequel add the name of the variable as an index when speaking of algebras of (extended or not) pseudodifferential symbols.

Definition 2.4 (non-local transformation Θ) *Let $\Theta : D\Psi D_\xi \to \Psi D_r$ be the associative algebra isomorphism defined by*

$$\begin{aligned}
&\partial_\xi^{\frac{1}{2}} \to \partial_r, \quad \partial_\xi^{-\frac{1}{2}} \to \partial_r^{-1} \\
&\xi \to \frac{1}{2} r \partial_r^{-1}, \quad \xi^{-1} \to 2\partial_r r^{-1} \qquad (2.5)
\end{aligned}$$

The inverse morphism $\Theta^{-1} : \partial_r \to \partial_\xi^{\frac{1}{2}}$, $r \to 2\xi\partial_\xi^{\frac{1}{2}}$ is easily seen to be an algebra isomorphism because the defining relation $[\partial_r, r] = 1$ is preserved by Θ^{-1}. It may be seen formally as the integral transformation $\psi(r) \to \tilde{\psi}(\xi) := \int_{-\infty}^{+\infty} \frac{e^{-r2/4\xi}}{\sqrt{\xi}} \psi(r)\, dr$ (one verifies straightforwardly for instance that $r\partial_r\psi$ goes to $2\xi\partial_\xi\tilde{\psi}$ and that $\partial_r 2\psi$ goes to $\partial_\xi\tilde{\psi}$). In other words, assuming $\psi \in L1(\mathbb{R})$, one has $\tilde{\psi}(\xi) = (P_\xi\psi)(0)$ $(\xi \geq 0)$ where $(P_\xi, \xi \geq 0)$ is the usual heat semi-group. Of course, this does not make sense at all for $\xi < 0$.

Remark. Denote by $\mathcal{E}_r = [r\partial_r, .]$ the Euler operator. Let $\Psi D_{(0)}$, resp. $\Psi D_{(1)}$ be the vector spaces generated by the operators $D \in \Psi D$ such that $\mathcal{E}_r(D) = nD$ where n is even, resp. odd. Then $\Psi D_{(0)}$ is an (associative) subalgebra of ΨD, and one has

$$[\Psi D_{(0)}, \Psi D_{(0)}] = \Psi D_{(0)}, \quad [\Psi D_{(0)}, \Psi D_{(1)}] = \Psi D_{(1)}, \quad [\Psi D_{(1)}, \Psi D_{(1)}] = \Psi D_{(0)}.$$

Now, the inverse image of $D \in \Psi D_r$ by Θ^{-1} belongs to $\Psi D_\xi \subset D\Psi D_\xi$ if and only if $D \in (\Psi D_r)_{(0)}$.

Lemma 2.5 (pull-back of Adler's trace) *The pull-back by Θ of Adler's trace on ΨD_r yields a trace on $D\Psi D$ defined by*

$$\mathrm{Tr}_{D\Psi D_\xi}(a(\xi)\partial_\xi^q) \ := \ Tr_{\Psi D_r}\left(\Theta(a(\xi)\partial_\xi^q)\right) = 2\delta_{q,-1} \cdot \frac{1}{2i\pi}\oint a(\xi)d\xi. \tag{2.6}$$

Proof.

Note first that the Lie bracket of ΨD_r, resp. $D\Psi D_\xi$ is graded with respect to the adjoint action of the Euler operator $\mathcal{E}_r := [r\partial_r, .]$, resp. $\mathcal{E}_\xi := [\xi\partial_\xi, .]$, and that $\Theta \circ \mathcal{E}_\xi = \frac{1}{2}\mathcal{E}_r \circ \Theta$. Now $\mathrm{Tr}_{\Psi D_r} D = 0$ if $D \in \Psi D_r$ is not homogeneous of degree 0 with respect to $\mathcal{E}_r$, hence the same is true for $\mathrm{Tr}_{D\Psi D_\xi}$. Consider $D := \xi^j\partial_\xi^j = \Theta^{-1}((\frac{1}{2}r\partial_r^{-1})^j\partial_r^{2j})$: then $\mathrm{Tr}_{D\Psi D_\xi}(D) = 0$ if $j \geq 0$ because (as one checks easily by an explicit computation) $\Theta(D) \in OD$; and $\mathrm{Tr}_{D\Psi D_\xi}(D) = 0$ if $j \leq -2$ because $\Theta(D) = O(\partial_r^{-2})$.

□

In order to obtain time-dependent equations, one needs to add an extra dependence on a formal parameter t of all the algebras we introduce. One obtains in this way loop algebras, whose formal definition is as follows:

Definition 2.6 (loop algebras) *Let $\mathfrak{g}$ be a Lie algebra. Then the* loop algebra *over $\mathfrak{g}$ is the Lie algebra*

$$\mathfrak{L}_t\mathfrak{g} := \mathfrak{g}[t, t^{-1}]]. \tag{2.7}$$

Elements of $\mathfrak{L}_t\mathfrak{g}$ may also be considered as Laurent series, or simply as functions $t \to X(t)$, where $X(t) \in \mathfrak{g}$.

The transformation Θ yields immediately (by lacing with respect to the time-variable t) an algebra isomorphism

$$\mathfrak{L}_t\Theta : \mathfrak{L}_t(D\Psi D_\xi) \to \mathfrak{L}_t(\Psi D_r), \quad D \to (t \to \Theta(D(t))). \tag{2.8}$$

3 Time-shift transformation and symmetries of the free Schrödinger equation

In order to define extended symmetries of the Schrödinger equation, one must first introduce the following time-shift transformation.

Definition 3.1 (time-shift transformation $\mathcal{T}_t$) *Let $\mathcal{T}_t : D\Psi D_\xi \to \mathfrak{L}_t(D\Psi D_\xi)$ be the linear transformation defined by*

$$\mathcal{T}_t\left(f(\xi)\partial_\xi^\kappa\right) = (\mathcal{T}_t f(\xi))\partial_\xi^\kappa \tag{3.1}$$

where:

$$\mathcal{T}_t P(\xi) = P(t+\xi) \tag{3.2}$$

for polynomials P, *and*

$$\mathcal{T}_t \xi^{-k} = (t+\xi)^{-k} := t^{-k} \sum_{j=0}^{\infty} (-1)^j \frac{k(k+1)\dots(k+j-1)}{j!} (\xi/t)^j. \tag{3.3}$$

In other words, for any Laurent series $f \in \mathbb{C}[\xi, \xi^{-1}]]$,

$$\mathcal{T}_t f(\xi) = \sum_{j=0}^{\infty} \frac{f^{(j)}(t)}{j!} \xi^j.$$

Then $\mathcal{T}_t$ is an injective Lie algebra homomorphism, with left inverse $\mathcal{S}_t$ given by

$$\mathcal{S}_t(g(t,\xi)) = \frac{1}{2\mathrm{i}\pi} \oint g(\xi, t) \frac{dt}{t}. \tag{3.4}$$

Proof. Straightforward.

□

Now comes an essential remark (see Introduction) which we shall first explain in an informal way. The free Schrödinger equation reads in the 'coordinates' (t, ξ)

$$(-2\mathrm{i}\mathcal{M}\partial_t - \partial_\xi)\tilde{\psi}(t,\xi) = 0. \tag{3.5}$$

In the complex coordinates $z = t - 2\mathrm{i}\mathcal{M}\xi$, $\bar{z} = t + 2\mathrm{i}\mathcal{M}\xi$, one simply gets (up to a constant) the $\bar{\partial}$-operator, whose algebra of Lie symmetries is $\mathrm{span}(f(t-2\mathrm{i}\mathcal{M}\xi)\partial_\xi, g(t+2\mathrm{i}\mathcal{M}\xi)\partial_t)$ for arbitrary functions f, g. An easy but crucial consequence of these considerations is the following:

Definition 3.2 ($\mathcal{X}_f^{(i)}$-generators) *Let, for $f \in \mathbb{C}[\xi,\xi^{-1}]]$ and $j \in \frac{1}{2}\mathbb{Z}$,*

$$\mathcal{X}_f^{(j)} = \Theta_t(-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi^j) \tag{3.6}$$

where Θ_t is the composition of the non-local transformation Θ and the time-shift $\mathcal{T}_t$,

$$\Theta_t := \mathfrak{L}_t(\Theta) \circ \mathcal{T}_t. \tag{3.7}$$

Lemma 3.3 (invariance of the Schrödinger equation) *The free Schrödinger equation $\Delta_0\psi(t,r) = 0$ is invariant under the Lie algebra of transformations generated by $\mathcal{X}_f^{(i)}$, $i \in \frac{1}{2}\mathbb{Z}$.*

Proof. Straightforward. □

The remarkable fact is that (denoting by $\dot{f}, \ddot{f}, \frac{d3f}{dt3}$ the time-derivatives of f of order 1, 2, 3)

$$\mathcal{X}_f^{(1)} = -f(t)\partial_r 2 + \mathrm{i}\mathcal{M}\dot{f}(t)r\partial_r + \frac{1}{2}\mathcal{M}2\ddot{f}(t)r2 - \left(\frac{1}{2}\mathcal{M}2\ddot{f}(t)r + \frac{\mathrm{i}}{6}\frac{d3f}{dt3}r3\right)\partial_r^{-1} + O(\partial_r^{-2}); \tag{3.8}$$

$$\mathcal{X}_g^{(\frac{1}{2})} = -g(t)\partial_r + \mathrm{i}\mathcal{M}\dot{g}(t)r + O(\partial_r^{-1}); \tag{3.9}$$

$$\mathcal{X}_h^{(0)} = -h(t) + O(\partial_r^{-1}). \tag{3.10}$$

In other words (up to constant multiplicative factors), the projection $(\mathcal{X}_f^{(k)})_+$ of $\mathcal{X}_f^{(k)}$, $k = 1, \frac{1}{2}, 0$ onto OD forms a Lie algebra which coincides with the realization $d\pi_0$ (see Definition 1.2), *apart* from the fact that $-2\mathrm{i}\mathcal{M}\partial_t$ is substituted by $\partial_r 2$ in the formula for $\mathcal{X}_f^{(1)}$. This discrepancy is not too alarming since $-2\mathrm{i}\mathcal{M}\partial_t \equiv \partial_r 2$ on the kernel of the free Schrödinger operator. As we shall see below, one may alter the $\mathcal{X}_f^{(1)}$ in order to 'begin with' $-f(t)\partial_t$ as expected, but then the $\mathcal{X}_f^{(1)}$ appear to have a specific definition.

4 Central cocycles of $(\Psi D_r)_{\le 1}$ and derivations

The above symmetry generators of the free Schrödinger equation, $\mathcal{X}_f^{(i)}$, $i \ge 1$ may be seen as elements of $\mathcal{L}_t(\Psi D_r)$. The original idea (following the scheme for Hill operators recalled in the Introduction) was to try and embed the space of Schrödinger operators $\mathcal{S}^{aff}$ into the dual of $\mathcal{L}_t(\Psi D_r)$ and realize the action $d\sigma_{1/4}$ of Proposition 1.4 as part of the coadjoint representation of an appropriate central extension of $\mathcal{L}_t(\Psi D_r)$.

Unfortunately this scheme is a little too simple: it allows to retrieve only the action of the Y- and M-generators, as could have been expected from the remarks at the end of section 3. It turns out that the $\mathcal{X}_f^{(i)}$, $i \leq \frac{1}{2}$ may be seen as elements of $\mathfrak{L}_t((\Psi D_r)_{\leq 1})$, while the realization $d\pi_0(\mathcal{L}_f)$ (see Definition 1.2) of the generators in $\mathrm{Vect}(S1) \subset \mathfrak{sv}$ involve *outer derivations* of this looped algebra. Then the above scheme works correctly, provided one chooses the right central extension of $\mathfrak{L}_t((\Psi D_r)_{\leq 1})$. As explained below, there are many possible families of central extensions, and the correct one is obtained by 'looping' a cocycle $c_3 \in H2((\Psi D_r)_{\leq 1}, \mathbb{R})$ which does *not* extend to the whole Lie algebra ΨD_r.

4.1 Central cocycles of $(\Psi D_r)_{\leq 1}$

We shall (almost) determine $H2(\Psi D_{\leq 1})$, using its natural semi-direct product structure: $\Psi D_{\leq 1} = \mathrm{Vect}(S1) \ltimes \Psi D_{\leq 0}$.

One has (either by using the Hochschild-Serre spectral sequence or by ???):

$$H2(\Psi D_{\leq 1}) = H2(\mathrm{Vect}(S1), \mathbb{R}) \oplus H1(\mathrm{Vect}(S1), H1(\Psi D_{\leq 0})) \oplus Inv_{\mathrm{Vect}(S1)} H2(\Psi D_{\leq 0}). \quad (4.1)$$

The one-dimensional space $H2(\mathrm{Vect}(S1), \mathbb{R})$ is generated by the Virasoro cocycle, which we shall denote by c_0.

For the second piece, elementary computations give $[\Psi D_{\leq 0}, \Psi D_{\leq 0}] = \Psi D_{\leq -2}$. So $H_1(\Psi D_{\leq 0})$ is isomorphic to $\Psi D_{\leq 0}/\Psi D_{\leq -2}$, i.e. to the space of symbols of type $f_0 + f_{-1}\partial^{-1}$. In terms of density modules, one has$H_1(\Psi D_{\leq 0}) = \mathcal{F}_0 \oplus \mathcal{F}_1$. So $H1(\Psi D_{\leq 0}) = (\mathcal{F}_0 \oplus \mathcal{F}_1)* = \mathcal{F}_{-1} \oplus \mathcal{F}_0$ by the usual duality, and $H1(\mathrm{Vect}(S1), H1(\Psi D_{\leq 0}) = H1(\mathrm{Vect}(S1), \mathcal{F}_{-1} \oplus \mathcal{F}_0) = H1(\mathrm{Vect}(S1), \mathcal{F}_{-1}) \oplus H1(\mathrm{Vect}(S1), \mathcal{F}_0)$. From the results of Fuks[], one knows that$H1(\mathrm{Vect}(S1), \mathcal{F}_1)$ is one-dimensional, generated by $f\partial \longrightarrow f''dx$, and$H1(\mathrm{Vect}(S1), \mathcal{F}_0)$ is two-dimensional, generated by $f\partial \longrightarrow fandf\partial \longrightarrow f'$. So we have proved that $H1(\mathrm{Vect}(S1), H1(\Psi D_{\leq 0})$ is three-dimensional with generators c_1, c_2 and c_3 as follows:

$$c_1(g\partial, \sum_{k=-\infty} 0 f_k \partial^k) = \int_{S^1} g^{''} f_0 \, dt \quad (4.2)$$

$$c_2(g\partial, \sum_{k=-\infty} 0 f_k \partial^k) = \int_{S^1} g f_{-1} \, dt \quad (4.3)$$

$$c_3(g\partial, \sum_{k=-\infty} 0 f_k \partial^k) = \int_{S^1} g^{'} f_{-1} \, dt \quad (4.4)$$

Let us finally consider the third piece $Inv_{\mathrm{Vect}(S1)} H2(\Psi D_{\leq 0})$.We shall once more make use of a decomposition into a semi-direct product: setting $Volt = \Psi D_{\leq -1}$, one has $\Psi D_{\leq 0} = \mathcal{F}_0 \ltimes$ Volt, where $\mathcal{F}_0$ is considered as an abelian Lie algebra, acting non-trivially on $Volt$. We do not know how to compute the cohomology of $Volt$, because of its "pronilpotent" structure, but we shall make the following:

Conjecture:

$$Inv_{\mathcal{F}_0} H2(Volt) = 0. \quad (4.5)$$

We shall now work out the computations modulo this conjecture. first one gets$H2(\Psi D_{\leq 0}) = H2(\mathcal{F}_0) \oplus H1(\mathcal{F}_0, H1(Volt))$

Then $Inv_{\mathrm{Vect}(S1)} H2(\Psi D_{\leq 0}) = Inv_{\mathrm{Vect}(S1)} H2(\mathcal{F}_0) \oplus Inv_{\mathrm{Vect}(S1)} H1(\mathcal{F}_0, H1(Volt))$. Since $\mathcal{F}_0$ is abelian, one has $H2(\mathcal{F}_0) = \Lambda 2(\mathcal{F}_0^*)$, and $Inv_{\mathrm{Vect}(S1)}(\Lambda 2(\mathcal{F}_0^*))$ is one-dimensional, generated by the well-known cocycle

$$c_4(f,g) = \int_{S^1} (g^{'} f - f^{'} g)\ dt. \tag{4.6}$$

A direct computation then shows that $[Volt, Volt] = \Psi D_{\leq -3}$, so $H_1(Volt) = \mathcal{F}_{-1} \oplus \mathcal{F}_{-2}$ and $H1(Volt) = \mathcal{F}_0 \oplus \mathcal{F}_1$ as $\mathrm{Vect}(S1)$-module. Then $H1(\mathcal{F}_0, H1(Volt))$ is easily determined by direct computation, as well as $Inv_{\mathrm{Vect}(S1)} H1(\mathcal{F}_0, H1(Volt))$; the latter is one-dimensional, generated by the following cocycle:

$$c_5(g, \sum_{k=-\infty} 0 f_k \partial^k) = \int_{S^1} g f_{-1}\ dt, \tag{4.7}$$

Let us summarize our results in the following:

Proposition 4.1 *Assuming conjecture (4.5) holds true, the space $H2(\Psi D_{\leq 1})$is six-dimensional, generated by the cocycles c_i, $i = 0, \ldots, 5$, defined above.*

Remarks:

1. If conjecture (4.5) turned out to be false, it could only add some supplementary generators; in any case, we have proved that $H2(\Psi D_{\leq 1})$ is at least six-dimensional.
2. The natural inclusion $i : \Psi D_{\leq 1} \longrightarrow \Psi D$ induces $i^* : H2\Psi D \longrightarrow H2(\Psi D_{\leq 1})$; one may then determine the image by i^* of the two generators of $H2(\Psi D)$ determined by Kravchenko and Khesin[]. Set $c_{KK_1}(D_1, D_2) = \int_{S^1} \kappa([Logr, D_1], D_2) dr$ and $c_{KK_2}(D_1, D_2) = \int_{S^1} \kappa([Log\partial, D_1], D_2) dr$. Then $i^* c_{KK_1} = c_2$ and $i^* c_{KK_2} = c_0 + c_1 + c_4$.

The right cocycle for our purposes turns out to be c_3: one gets a centrally extended Lie algebra of pseudodifferential symbols $\widetilde{\Psi D}_{\leq 1}$ as follows

Definition.

Let $\widetilde{\Psi D}_{\leq 1}$ be the central extension of $\Psi D_{\leq 1}$ associated with the cocycle $2c_3$.

4.2 Derivations of the looped centrally extended algebra

Let us introduce now the looped algebra $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})$ in order to allow for time-dependence.

An element of $\mathcal{L}(\widetilde{\Psi D}_{\leq 1})$ is a pair $(D(t),\lambda(t))$ where $\lambda \in \mathbb{C}[t,t^{-1}]]$ and $D(t) \in \mathfrak{L}_t((\Psi D_r)_{\leq 1})$. By a slight abuse of notation, we shall write $c_3(D_1,D_2)$ $(D_1,D_2 \in \mathfrak{L}_t((\Psi D_r)_{\leq 1}))$ for the function $t \to c_3(D_1(t),D_2(t))$, so now c_3 has to be seen as a function-valued central cocycle of $\mathfrak{L}_t((\Psi D_r)_{\leq 1})$. In other words, we consider the looped version of the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \widetilde{(\Psi D_r)_{\leq 1}} \longrightarrow (\Psi D_r)_{\leq 1} \longrightarrow 0, \tag{4.8}$$

namely,

$$0 \longrightarrow \mathbb{R}[t,t^{-1}]] \longrightarrow \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}}) \longrightarrow \mathfrak{L}_t((\Psi D_r)_{\leq 1}) \longrightarrow 0. \tag{4.9}$$

Lemma 4.2 (derivations of $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})$) *The Lie algebra $Der(\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}}))$ of Lie derivations of $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})$ includes the following outer derivations:*

(i) time-reparametrizations $f(t)\partial_t : (D(t),\lambda(t)) \to (f(t)\dot{D}(t), f(t)\dot{\lambda}(t))$;

(ii) time-dependent Euler operators $f(t)\mathcal{E}_r$ defined by:

$$f(t)\mathcal{E}_r\,(g(t)r^p(r\partial_r)^q),\lambda(t)) = (f(t)g(t)\ .\ pr^p(r\partial_r)^q, 0)\,. \tag{4.10}$$

Proof.

(i) is not specific of $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})$ (such infinitesimal time reparametrizations may be considered for any loop algebra), so let us concentrate on (ii). If one forgets first about the central extension, $f(t)\mathcal{E}_r$ may be identified with the adjoint action of $f(t)r\partial_r$ which is an inner derivation. So one must only check that the action of $f(t)\mathcal{E}_r$ is compatible with the central extension, namely,

$$(0, c_3\,(f(t)\mathcal{E}_r(g_1(t)r^{p_1}\partial_r^{q_1}), g_2(t)r^{p_2}\partial_r^{q_2}) - c_3\,(f(t)\mathcal{E}_r(g_2(t)r^{p_2}\partial_r^{q_2}), g_1(t)r^{p_1}\partial_r^{q_1}))\; = 0. \tag{4.11}$$

This identity is trivial except if $q_1 = -1$, $q_2 = 1$. Then

$$c_3\left(f(t)\mathcal{E}_r(g_1(t)r^{p_1}\partial_r^{-1}), g_2(t)r^{p_2}\partial_r\right) = (p_1+1)fg_1g_2\ .\ p_1\delta_{p_1+p_2,0} \tag{4.12}$$

and

$$c_3\left(f(t)\mathcal{E}_r(g_2(t)r^{p_2}\partial_r), g_1(t)r^{p_1}\partial_r^{-1}\right) = (p_2-1)fg_1g_2\ .\ p_2\delta_{p_1+p_2,0} \tag{4.13}$$

hence the left-hand side of (4.11) is 0.

□

5 I-embedding of $(D\Psi D_\xi)_{\leq 1}$ into the looped algebra $\mathfrak{g}$

This section, as explained in the introduction to section 4, is devoted to the construction of an explicit embedding (called: *I-embedding*) of the abstract algebra of extended pseudodifferential symbols $(D\Psi D_\xi)_{\leq 1}$ into a Lie algebra $\mathfrak{g}$ which is a semi-direct product, $\mathfrak{g} \simeq \mathrm{Vect}(S1) \ltimes \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})$. Loosely speaking, the image $I((D\Psi D_\xi)_{\leq 1})$ is made up of the $\mathcal{X}_f^{(j)}$, $j \leq \frac{1}{2}$ and the $\mathcal{X}_f^{(1)}$ *with* $\partial_r 2$ *substituted by* $-2\mathrm{i}\mathcal{M}\partial_t$ (see end of section 3).

Theorem 5.1 (homomorphism I) *Let* $I : (D\Psi D_\xi)_{\leq 1} \simeq \mathrm{Vect}(S1)_\xi \ltimes (D\Psi D_\xi)_{\leq \frac{1}{2}} \hookrightarrow Der(\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})) \ltimes \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})$ *be the mapping defined by*

$$(-\frac{\mathrm{i}}{2\mathcal{M}} f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi, D) \to (-f(t)\partial_t - \frac{1}{2}\dot{f}(t)\mathcal{E}_r, \frac{\mathrm{i}}{2\mathcal{M}}(\Theta_t(-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi))_{\leq 0} + \Theta_t(D)) \quad (5.1)$$

where by definition $\left(\sum_{k=-\infty}^{N} f(t,r)\partial_r^k\right)_{\leq 0} := \sum_{k=-\infty} 0 f(t,r)\partial_r^k$.

Then I is a Lie algebra homomorphism.

Remark:

Note first that $\mathcal{L}'_f := -f(t)(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r 2)$ is an independent copy of $\mathrm{Vect}(S1)$, by which we mean that $[\mathcal{L}'_f, \mathcal{L}'_g] = \mathcal{L}'_{\{f,g\}} = \mathcal{L}'_{fg'-f'g}$ and $[\mathcal{L}'_f, \mathcal{X}_f^{(i)}] = 0$ for all i. This is immediate in the 'coordinates' (t,ξ) since $\Theta(\mathcal{L}'_f) = -f(t)(-2\mathrm{i}\mathcal{M}\partial_t - \partial_\xi)$ commutes trivially with $\Theta(\mathcal{X}_f^{(i)}) = -f(t - 2\mathrm{i}\mathcal{M}\xi)\partial_\xi^i$. If one leaves aside the second coordinate in $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})$, i.e. the central extension, then $I(-\frac{\mathrm{i}}{2\mathcal{M}} f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi)$ may be identified with

$$\frac{\mathrm{i}}{2\mathcal{M}}(\mathcal{L}'_f + \mathcal{X}_f^{(1)}) \equiv -f(t)\partial_t - \frac{1}{2}\dot{f}(t)r\partial_r + \frac{\mathrm{i}\mathcal{M}}{4}\dddot{f}(t)r2 + \ldots$$

which coincides with $d\pi_0(\mathcal{L}_f)$ (see Definition 1.2). Hence (up to the central extension once again), $I|_{\mathrm{Vect}(S1)_\xi \times \{0\}}$ is a homomorphism. Moreover, the $I(-\frac{\mathrm{i}}{2\mathcal{M}} f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi)$ have the same action on the $\mathcal{X}_f^{(i)}$, $i \in \frac{1}{2}\mathbb{Z}$ as

$$\frac{\mathrm{i}}{2\mathcal{M}}\mathcal{X}_f^{(1)} = \frac{\mathrm{i}}{2\mathcal{M}}\Theta_t(-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi) = -f(t)\partial_r 2 - \frac{1}{2}\dot{f}(t)r\partial_r + + \frac{\mathrm{i}\mathcal{M}}{4}\dddot{f}(t)r2 + \ldots$$

Proof of Theorem 5.1

Forgetting about the central extension, the Theorem is true by the above remark.

Now $c_3(\Theta_t(D_1), \Theta_t(D_2))) = 0$ if $D_1, D_2 \in (D\Psi D_\xi)_{\leq \frac{1}{2}}$ since $D_j = f_j(t) + O(\partial_r 0)$, $j = 1, 2$, hence $I|_{\{0\}\times(D\Psi D_\xi)_{\leq \frac{1}{2}}}$ is a homomorphism.

Similarly, $c_3(\Theta_t(D_1),D)=0$ if $D\in\mathfrak{L}_t((\Psi D_r))_{\le 0}$. Note also that (letting $\rho_0(\mathcal{L}_f):=-f(t)\partial_t-\frac{1}{2}\dot{f}(t)\mathcal{E}_r\in Der(\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})))$ one has plainly $\rho_0(\mathcal{L}_{\{f,g\}})=[\rho_0(\mathcal{L}_f),\rho_0(\mathcal{L}_g)]$ (see Lemma 4.2). Hence (letting $\rho(\mathcal{L}_f)=I(-\frac{\mathrm{i}}{2\mathcal{M}}f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi))$

$$\begin{aligned}[\rho(\mathcal{L}_f),I(D)]_{Der(\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}}))\ltimes\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})} &= [\frac{\mathrm{i}}{2\mathcal{M}}(\mathcal{L}'_f+\mathcal{X}_f^{(1)}),\Theta_t(D)]_{\Psi D_{t,r}} \\ &= [d\pi_0(\mathcal{L}_f),\Theta_t(D)]_{\Psi D_{t,r}} \\ &= \frac{\mathrm{i}}{2\mathcal{M}}[\mathcal{X}_f^{(1)},\Theta_t(D)]_{\Psi D_{t,r}} \qquad (5.2)\end{aligned}$$

(see remarks before the proof) if $D\in(D\Psi D_\xi)_{\le\frac{1}{2}}$, where $[\ .\ ,\ .\ ]_{\Psi D_{t,r}}$ stands for the usual Lie bracket of pseudo-differential symbols in *two* variables.

Similarly,

$$\begin{aligned}&[\rho(\mathcal{L}_f),\rho(\mathcal{L}_g)]_{Der(\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}}))\ltimes\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})}=\Big(\rho_0(\mathcal{L}_{\{f,g\}}),\quad \rho_0(\mathcal{L}_f).\frac{\mathrm{i}}{2\mathcal{M}}\left(\Theta_t(-g(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi)\right)_{\le 0} \\ &-\rho_0(\mathcal{L}_g).\frac{\mathrm{i}}{2\mathcal{M}}\left(\Theta_t(-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi)\right)_{\le 0}+\left(\frac{\mathrm{i}}{2\mathcal{M}}\right)2\left[(\Theta_t(-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi))_{\le 0},(\Theta_t(-g(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi))_{\le 0}\right]_{\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})}\Big) \\ &=\Big(\rho_0(\mathcal{L}_{\{f,g\}}),\left[-f(t)\partial_t-\frac{1}{2}\dot{f}(t)r\partial_r,\frac{\mathrm{i}}{2\mathcal{M}}\left(\Theta_t(-g(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi)\right)_{\le 0}\right]_{\Psi D_{t,r}}-\Big[-g(t)\partial_t-\frac{1}{2}\dot{g}(t)r\partial_r, \\ &\frac{\mathrm{i}}{2\mathcal{M}}\left(\Theta_t(-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi)\right)_{\le 0}\Big]_{\Psi D_{t,r}}+\left(\frac{\mathrm{i}}{2\mathcal{M}}\right)2\left[(\Theta_t(-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi))_{\le 0},(\Theta_t(-g(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi))_{\le 0}\right]_{\Psi D_{t,r}}\Big) \\ &=\rho(\mathcal{L}_{\{f,g\}}). \qquad (5.3)\end{aligned}$$

□

Definition 5.1 (ρ-action of $Vect(S1)$ on $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$) *Let* $\mathrm{Vect}(S1)_t$ *be the image by I in* $Der(\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}}))\ltimes\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ *of the Lie subalgebra* $(\mathrm{Vect}(S1)_\xi,0)$. *We denote by*

$$\rho:\mathrm{Vect}(S1)_t\to\mathfrak{gl}(Der(\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}}))\ltimes\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})) \qquad (5.4)$$

the adjoint action of $\mathrm{Vect}(S1)_t$ *on* $Der(\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}}))\ltimes\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$. *Hence one has an embedding of the semi-direct product* $\mathrm{Vect}(S1)_t\ltimes_\rho\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ *into* $Der(\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}}))\ltimes\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$.

In other words, the mapping I factorizes through a mapping $I':(D\Psi D_\xi)_{\le 1}\hookrightarrow\mathrm{Vect}(S1)_t\ltimes_\rho\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$.

We may now finally introduce the Lie algebra $\mathfrak{g}$ announced in the Introduction (see also introduction to section 4), together with the I-embedding $(D\Psi D_\xi)_{\le 1}\hookrightarrow\mathfrak{g}$.

Definition 5.2 *Let* $\mathfrak{g}_0=\mathrm{Vect}(S1)_t$, $\mathfrak{h}=\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ *and* $\mathfrak{g}:=\mathfrak{g}_0\ltimes_\rho\mathfrak{h}$, *so I' maps* $(D\Psi D_\xi)_{\le 1}$ *into* $\mathfrak{g}$.

As we shall see in the next two sections, the coadjoint representation of the semi-direct product $\mathfrak{g}$ is the key to define a symplectic structure on $\mathcal{S}^{aff}$ for which the action of SV is Hamiltonian.

6 Projected coadjoint action of $\mathfrak{g}$

Recall first the following easy lemma (see [5], Lemma 3.1)

Lemma 6.1 *Let $\mathfrak{g} = \mathfrak{g}_0 \ltimes \mathfrak{h}$ be a semi-direct product of two Lie algebras $\mathfrak{g}_0$ and $\mathfrak{h}$. Then the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is given by*

$$\mathrm{ad}^*_{\mathfrak{g}}(L,D).(\lambda,\delta) = \langle \mathrm{ad}^*_{\mathfrak{g}_0}(L)\lambda - \tilde{D}.\delta, \tilde{L}^*(\delta) + \mathrm{ad}^*_{\mathfrak{h}}(D).\delta\rangle$$

where by definition

$$\langle \tilde{D}.\delta, L\rangle_{\mathfrak{g}_0^*\times\mathfrak{g}_0} = \langle \tilde{L}^*(\delta), D\rangle_{\mathfrak{h}^*\times\mathfrak{h}} = \langle \delta, [L,D]\rangle_{\mathfrak{h}^*\times\mathfrak{h}}.$$

Proof. Straightforward. □

Let $\pi_1 : \mathfrak{h} \to \mathfrak{g}$ be the canonical Lie algebra embedding and $\pi_1^* : \mathfrak{g}^* \to \mathfrak{h}^*$ the dual linear Poisson morphism.

The coadjoint action $\mathrm{ad}^*_{\mathfrak{g}}$: $\mathfrak{g}^* \to \mathfrak{g}^*$ defines a Poisson morphism for the canonical KKS (Kirillov-Kostant-Souriau) structure on $\mathfrak{g}^*$. By composing with the Poisson morphism $\pi_1^* : \mathfrak{g}^* \to \mathfrak{h}^*$ and restricting to $\mathfrak{h}^*$, one obtains the 'projected' coadjoint action

$$\pi_1^* \circ \mathrm{ad}^*_{\mathfrak{g}} \;:\quad (L,D) \;:\; \delta \to \tilde{L}^*(\delta) + \mathrm{ad}^*_{\mathfrak{h}}(D).\delta \tag{6.1}$$

which is a Poisson morphism for the KKS structure on $\mathfrak{h}^*$.

We shall consider below the projected coadjoint action $\pi_1^* \circ \mathrm{ad}^*_{\mathfrak{g}}$ corresponding to the Lie algebra $\mathfrak{g}_0 \simeq \mathrm{Vect}(S1) \ltimes \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})$ introduced in Definition 5.2.

7 The action of $\mathfrak{sv}$ on Schrödinger operators as a projected coadjoint action

This section is devoted to the proof of the main Theorem announced in the Introduction, which we recall here:

Theorem 7.1 *The restriction of the projected coadjoint action $\pi_1^* \circ \mathrm{ad}^*_{\mathfrak{g}}|_{\mathfrak{sv}}$ on the submanifold $\{(V(t,r)\partial_r^{-2}, 1)\} \subset \mathfrak{h}^*$ coincides with the infinitesimal action $d\sigma_{1/4}$ of $\mathfrak{sv}$ on $\mathcal{S}^{aff} = \{-2\mathrm{i}\mathcal{M}\partial_t - \partial_r 2 + V(t,r)\}$.*

Proof of the Theorem.

Since an element of $\mathfrak{h} = \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\leq 1}})$ writes $(W(t), \lambda(t))$ with $W(t) \in (\Psi D_r)_{\leq 1}$, it is natural (using Adler's trace) to represent an element of the restricted dual $\mathfrak{h}^*$ as a couple $(D(t)dt, h(t)dt)$ with $D \in \mathfrak{L}_t((\Psi D_r)_{\geq -2})$ and $h \in C^\infty(S1)$. The coupling between $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ writes then

$$\langle (D(t)dt, h(t)dt)\, , (W(t), \lambda(t)) \rangle_{\mathfrak{h}^* \times \mathfrak{h}} = \frac{1}{2\mathrm{i}\pi} \oint \left(\mathrm{Tr}_{\Psi D_r}(D(t)W(t)) + h(t)\lambda(t)\right)\, dt. \tag{7.1}$$

The first important remark is that the projected coadjoint action $\pi_1^* \circ \mathrm{ad}^*$ of $(D\Psi D_\xi)_{\leq 1}$ on a homogeneous element $(V(t,r)\partial_r^{-2} dt, h(t)dt) \in \mathfrak{h}^*$ of degree -2 quotients out onto an action of the Schrödinger-Virasoro group: namely, let $\kappa \leq -\frac{1}{2}$ and $(W, \lambda) = (\sum_{j \leq 1} W_j(t,r)\partial_r^j, \lambda(t)) \in \mathfrak{h}$,

$$\begin{aligned}
&\langle \pi_1^* \circ \mathrm{ad}^*_{f(\xi)\partial_\xi^{-\kappa}} (V(t,r)\partial_r^{-2} dt, h(t)dt), W \rangle \\
&= \langle (V(t,r)\partial_r^{-2} dt, h(t)dt), \left[f(t)\partial_r^{-2\kappa} + O(\partial_r^{-2\kappa-1}), \sum_{j \leq 1} W_j(t,r)\partial_r^j \right]_{\mathfrak{h}} \rangle \\
&= 0
\end{aligned} \tag{7.2}$$

since the Lie bracket on the right-hand side produces a pseudodifferential operator of degree ≤ -1 and no central charge.

Let us now study successively the projected coadjoint action of the Y, M and L generators of $\mathfrak{sv}$ on homogeneous elements $(V(t,r)\partial_r^{-2} dt, h(t)dt) \in \mathfrak{h}^*$ of degree -2.

Recall from the Introduction that the derivative with respect to r, resp. t is denoted by $'$, resp. by a dot, namely, $V'(t,r) := \partial_r V(t,r)$ and $\dot{V}(t,r) := \partial_t V(t,r)$.

<u>Action of the Y-generators</u>

Let $W = \sum_{j \leq 1} W_j(t,r)\partial_r^j \in \mathfrak{L}_t((\Psi D_r)_{\leq 1})$ as before. A computation gives

$$\begin{aligned}
&\langle \pi_1^* \circ \mathrm{ad}^*_{\mathcal{Y}_g} (V(t,r)\partial_r^{-2} dt, h(t)dt), W \rangle = \\
&\langle (V(t,r)\partial_r^{-2} dt, h(t)dt), \Big[g(t)\partial_r + \frac{1}{2}\dot{g}(t)r + \frac{1}{8}\ddot{g}(t)r2\partial_r^{-1} + O(\partial_r^{-2}), \\
&W_1(t,r)\partial_r + O(\partial_r 0)\Big]_{\mathfrak{h}} \rangle \\
&= \langle (V(t,r)\partial_r^{-2} dt, h(t)dt), (g(t)W_1'(t,r)\partial_r + O(\partial_r 0), \frac{1}{2}\ddot{g}(t) \cdot \frac{1}{2\mathrm{i}\pi} \oint rW_1(r)dr) \rangle
\end{aligned} \tag{7.3}$$

Generally speaking (by definition of the duality given by Adler's trace), the terms in the above expression that depend on W_i, $i = 1, 0, \ldots$ give the projection of $\mathrm{ad}^*_{\mathcal{Y}_g}(Ddt, h(t)dt)$ on the component ∂_r^{-i-1}.

Hence altogether one has proved:

$$\pi_1^* \circ \mathrm{ad}^*_{\mathcal{Y}_g}((V(t,r)\partial_r^{-2},1)) = \left((-g(t)V'(t,r) + \frac{1}{2}\ddot{g}(t)r)\partial_r^{-2}, 0 \right) \tag{7.4}$$

as expected.

Action of the M-generators

It may be deduced from that of the Y-generators since the Lie brackets of the Y-generators generate all M-generators.

Action of the Virasoro part

Let $\rho_0(\mathcal{L}_f) := f(t)\partial_t + \frac{1}{2}\dot{f}(t)\mathcal{E}_r$ denote the action of $\mathrm{Vect}(S1)_t$ on $\mathfrak{h}$ by derivation as in Theorem 5.1.

One computes:

$$\begin{aligned}
&\langle \pi_1^* \circ \mathrm{ad}^*_{\mathcal{L}_f}(Ddt, h(t)dt), W\rangle = \\
&\langle (V(t,r)\partial_r^{-2}dt, h(t)dt), \rho_0(\mathcal{L}_f)\left(W_1(t,r)\partial_r + O(\partial_r 0)\right) \\
&+ \left[\frac{1}{8}r2\ddot{f}(t) + \left(-\frac{1}{8}\ddot{f}(t)r + \frac{1}{48}\frac{d3f}{dt3}r3\right)\partial_r^{-1}, W_1(t,r)\partial_r + O(\partial_r 0)\right]_{\mathfrak{h}}\rangle \\
&= \langle \big((V(t,r)\partial_r^{-2}dt, h(t)dt\big), \\
&(f(t)\dot{W}_1 + \frac{1}{2}\dot{f}(t)(rW_1' - W_1))\partial_r + O(\partial_r 0), \frac{1}{2\mathrm{i}\pi}\left(-\frac{1}{4}\ddot{f}(t)\oint W_1 dr + \frac{1}{8}\frac{d3f}{dt3}\oint r2W_1 dr\right)\rangle
\end{aligned} \tag{7.5}$$

Hence:

$$\pi_1^* \circ \mathrm{ad}^*_{\mathcal{L}_f}(V(t,r)\partial_r^{-2}) = \left(-f(t)\dot{V} - \frac{1}{2}\dot{f}(t)(rV' + 2V) - \frac{1}{4}\ddot{f}(t) + \frac{1}{8}r2\frac{d3f}{dt3}\right)\partial_r^{-2} \tag{7.6}$$

as expected.

References

[1] L. Guieu, C. Roger. *L'Algèbre et le Groupe de Virasoro: aspects géométriques et algébriques, généralisations.* Publications CRM, Montreal (2007).

[2] M. Henkel. *Schrödinger invariance and strongly anisotropic critical systems*, J. Stat. Phys. **75**, 1023 (1994).

[3] M. Henkel, J. Unterberger. *Supersymmetric extensions of Schrödinger invariance*, Nucl. Phys. **B746**, 155–201 (2006).

[4] V. Ovasienko, C. Roger. *Generalizations of Virasoro group and Virasoro algebra through extensions by modules of tensor-densities on* $S1$, Indag. Mathem., N.S., 9 (2), 277-288.

[5] C. Roger, J. Unterberger, *The Schrödinger-Virasoro Lie group and algebra: representation theory and cohomological study*. Ann. Henri Poincaré 7 (2006), 1477-1529.

[6] J. Unterberger, Journal of Physics, Conference Series **40**, 156 (2006).

[7] J. Unterberger. A classification of periodic time-dependent generalized harmonic oscillators using a Hamiltonian action of the Schrödinger-Virasoro group, preprint arXiv:0806.1185 (2008).